Note on the structure of constraint algebras

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Abstract

It is shown that when the gauge algebra is with root system the canonical Hamiltonian commutes with the constraints. Two other simple propositions concerning gauge fixing are proved too.

There is a vast literature on systems with constraints. Here we follow the Hamiltonian approach in which any constraint system is characterized by its Hamiltonian H and constraints $\varphi_a = 0,\ \ a = 1, \dots, m$ (all of them functions in the phase space). A thorough treatment of the subject could be found in [1]. In the present paper we shall consider the simplest and common case of order one models with first class (Bose) constraints. These requirements mean that the following Poisson bracket relations hold:

$$[\varphi_a, \varphi_b] = C_{abc}\varphi_c, \tag{1}$$
$$[H, \varphi_a] = U_{ab}\varphi_b, . \tag{2}$$

with coefficients U_{ab} and C_{abc} independent of dynamical variables (order one requirement). The coefficients C_{abc} are the structure constant of the algebra of gauge symmetry $\mathcal{A}$ in the model. As we see from eqs.(1) the generators of this algebra are the first class constraints φ.

A note should be added at this point. In the Hamiltonian approach the Hamiltonian is not uniquely determined. It is always possible to add to it a combination of constraints with arbitrary coefficients (these are the so called weakly zero terms). The origin of this ambiguity is the fact that in any calculation only the total Hamiltonian H_t

$$H_t = H + \lambda_a \varphi_a, \tag{3}$$

appears. Here λ_a are new (arbitrary) variables — the Lagrange multipliers. Formula (3) shows that any weakly zero term in the Hamiltonian can be absorbed in the $\lambda_a \varphi_a$ term after redefinition (with unity Jacobian) of the Lagrange multipliers. So, it is convenient to introduce the so called canonical Hamiltonian H_c which is Hamiltonian of the system with all weakly zero terms removed:

$$H_c = H|_{\varphi=0} \tag{4}$$

As usual we suppress the subscript 'c' in H_c and throughout the whole paper (including eqs.(2)) H denotes the canonical Hamiltonian.

Our aim here is to prove three propositions:

Proposition 1 If the gauge algebra $\mathcal{A}$ defined by eqs. (1) is a Lie algebra with root system, then the coefficients U_{ab} in eq.(2) are zeros.

Proof

Eqs.(2) originate from the requirement that the time evolution preserves the gauge algebra $\mathcal{A}$ generated by constraints φ. But it also asserts that φ and H form closed algebra which we shall denote by $\hat{\mathcal{A}}$.

There are two possibilities for the algebra $\hat{\mathcal{A}}$:

- it can coincide with the gauge algebra $\mathcal{A}$ (or with its universal envelope), or
- it can be with one generator larger.

In the first case $\hat{\mathcal{A}}$ is trivially closed. However, in this case H has to be a combination of the generators φ_a, and so is in fact zero because all weakly zero terms are removed from the canonical Hamiltonian H. Therefore, in this case $U_{ab} = 0$.

Consider now the case when $\hat{\mathcal{A}}$ is larger than the gauge algebra. Here we shall use the requirement that $\mathcal{A}$ is a Lie algebra with root system. For such algebra there are three possibilities for H. It can be

- a new independent step operator, or
- a new independent Cartan operator, or
- an independent central element of $\mathcal{A}$.

However, $\hat{\mathcal{A}}$ cannot be closed if H is a new step operator — there has to be at least one more step operator (the conjugated one) and also the corresponding Cartan element, both of which cannot be part of the gauge algebra (and thus of $\hat{\mathcal{A}}$ also). Same arguments show that $\hat{\mathcal{A}}$ is not closed if H is an independent Cartan operator. In this case we have at least one new root, so we have to add also at least one couple of new, independent step operators. The only possibility we are left with is that H is in the center of $\mathcal{A}$. But this means that $U_{ab} = 0$. **QED**

A proper treatment of any gauge model requires supplementary gauge fixing conditions. The only requirement on these conditions is their gauge noninvariance for any gauge transformation. Our next propositions concern two possible gauge choices — first, when gauge conditions are functions of the phase space variables and second, when they depend on Lagrange multipliers.

Proposition 2 If the gauge conditions depend on phase space variables only it is possible to add weakly zero terms to the canonical Hamiltonian, such that the new Hamiltonian has zero Poisson bracket with the gauge conditions.

Proof
The gauge transformation of any function in phase space is generated by constraints through the Poisson bracket relations. Therefore, functions χ_a of canonical coordinates and their momenta can be used as gauge conditions provided the operator Δ

$$\Delta_{ab} \equiv [\chi_a, \varphi_b] \tag{5}$$

is invertible and hereafter we shall suppose that this is fulfilled.

Let us introduce notations $y_a \equiv [H, \chi_a]$ and $D_{ab} : \quad D_{ab} f_b = \varphi_b \, [\chi_a, f_b]$. Our task is for $y \neq 0$ to find a vector α, such that

$$[H + \alpha_a \varphi_a, \chi_b] = 0. \tag{6}$$

A particular solution of this system of first order differential equations can be found as a formal series:

$$\begin{aligned} \alpha &= \sum_{n=0} \alpha^{(n)} \\ \alpha^{(0)} &= \Delta^{-1} y \\ \alpha^{(n)} &= -\Delta^{-1} \left[\chi, \alpha^{(n-1)} \right] \varphi. \end{aligned} \tag{7}$$

The solution given by eqs.(7) represents the series decomposition of $(\Delta + D)^{-1}$ in the neighborhood of the "point" Δ. **QED**

Proposition 3 The most general form of gauge condition, involving the Lagrangian multipliers only is $\lambda_a = \lambda^0_a$, where λ^0_a are some constants.

Proof
Suppose that we have a vector function f of λ such that $f(\lambda) = 0$ fixes gauge freedom completely, i.e.

$$\delta_\epsilon f \neq 0 \quad \forall \epsilon \neq 0. \tag{8}$$

Here δ_ϵ is the operator of infinitesimal gauge transformation with parameter ϵ. However, any variation of f is given by:

$$\delta f = \frac{\partial f}{\partial \lambda} \delta\lambda. \tag{9}$$

Provided that λ_a = constants, is a good gauge condition (which is usually supposed), then the operator of gauge transformations acting on the Lagrange multipliers has no zero modes and so, eq.(8) is fulfilled only provided the matrix $\frac{\partial f}{\partial \lambda}$ also has no zero modes. Therefore, $\dim(f) = m$ and

$$\det|\frac{\partial f}{\partial \lambda}| \neq 0 \tag{10}$$

in the vicinity of the point λ^0 such that $f(\lambda^0) = 0$. As a result $f(\lambda)$ are invertable functions. Their inverse we shall denote by $\bar{\lambda}(f)$. The expression

$$\lambda = \bar{\lambda}(0) = \lambda^0 \tag{11}$$

gives the equivalent transcription of the gauge condition $f(\lambda) = 0$. **QED**

References

[1] M Henneaux, Phys. Rep. **126** (1985) 1.